Cylindrically Symmetric Scalar Field and it's Lyapunov stability in General Relativity

H.R. Rezazadeh

Abstract In this paper we found an Exact solution for massless scalar field with cosmological constant. This exact solution generalized the Levi-Civita vacuum solution Levi-Civita (Rend. Acc. Lincei 27:183, 1917) to a massless scalar field, with a cosmological constant term.This solution in the absence of the Cosmological constant recovers the spacetime of a massless scalar field with cylindrical symmetry (*Buchdahl metric* (Buchdahl in Phys. Rev. 115:1325, 1959)). Also if the scalar field disappears, the spacetime will be a representation of de-Sitter space.We prove that the form of the metric's function which was purposed in Momeni and Miraghaei (Int. J. Mod. Phys. A 24(31):5991, 2009) is valid even if we assume a general form. Furthermore we show that in which conditions this solution satisfies energy conditions. Finally the credibility of focusing theorem is proved.

Keywords Gravitation · Exact solutions

1 Introduction

In the [1] the authors acclaimed that they were succeed in solving the problem of finding an exact solution of Einstein field equations for a massless scalar field with cylindrical symmetry in the presence of a cosmological constant [1]. They obtained a new two parameters exact solution (LB meric)[1] which recovers at least one member of Buchdahl family [2] in the absence of Cosmological constant and also $LC\Lambda$ family [3]. Their work in [1] completed previous works as [3, 6, 8, 12]. In that work the authors solved the field equations in a special case and nothing was stated about the general exact solution. Now we applied a method for solving field equations and show that we can construct a general two parameters exact solution which in the absence of the cosmological constant recover Buchdahl

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[1]This is an abbreviation to the memories of H.A. Buchdahl and the presence of the cosmological constant term.

H.R. Rezazadeh (✉)
Department of Mathematics, Faculty of Science, Islamic Azad University, Karaj branch, P.O. Box 31485-313, Karaj, Iran
e-mail: h-rezazadeh@kiau.ac.ir

solution. We show that this system of differential equations poses only those solutions that the authors stated in [1]. Later a class of solutions of Einstein field equations is investigated for a cylindrically symmetric spacetime when the source of gravitation is a perfect fluid [4]. As a historical note we added here that, our solution with Cosmological constant and scalar field is a generalization of Levi-Civita family [7, 8]. Previously the exact solution of Einstein field equations with a cosmological constant term were found by Linet [11] and in a more efficient form by Tian [12]. The singularity problem in a family of cylindrically symmetric spacetimes which is described by the collapsing of scalar fields was discussed by Wang and Frankel [9, 10]. Senovilla presented an explicit exact solution of Einstein's equations for an inhomogeneous dust universe with cylindrical symmetry [5]. In this work at first we prove that the general solution for field equations only is one, which was stated in [1]. Then we investigate the validity of Energy conditions and focusing theorem for our solution. Also a short section is presented which discusses the stability of exact solution under small perturbations via Lyapunov exponents method [50–52].

2 Field Equations

We begin with a general cylindrically symmetric metric in Weyl coordinates (t, r, φ, z),

$$ds^2 = -e^{u(r)}dt^2 + dr^2 + e^{v(r)}d\varphi^2 + e^{w(r)}dz^2 \tag{1}$$

Field equation for a massless minimally coupled scalar field in the presence of a cosmological constant term Λ is reading as:[2]

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} \tag{2}$$

We labeled metric functions as $u_i = \{u(r), v(r), w(r)\}$, $\phi \equiv \phi(r)$, $\acute{h} = \frac{dh}{dr}$. In terms of these functions we can rewrite field equation (2) in the following succinct forms:

$$2u_i'' + u_i' \sum_{j=1}^{3} u_j' - 4\Lambda = 0, \quad i = \{1, 2, 3\} \tag{3}$$

$$2\sum_{j=1}^{3} u_j'' + \sum_{j=1}^{3} u_j'^2 - 4\Lambda = 4\phi'^2 \tag{4}$$

Now we write (3) in the following simple form:

$$\frac{d}{dr}(u_i' e^f) = 2\Lambda e^f, \quad i = \{1, 2, 3\} \tag{5}$$

where in it $f = \frac{\sum_{j=1}^{3} u_j}{2}$. If we do summation on $i = 1, 2, 3$ in (5) we can write a differential equation for new function $f = f(r)$,

$$\frac{d}{dr}(f' e^f) = 3\Lambda e^f \tag{6}$$

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[2]We will mostly use natural units $\hbar = c = 1$ and $8\pi G = 1$.

In terms of this new function (4) converted to the following equation:

$$3f'' + f'^2 - 7\Lambda = \phi'^2 \tag{7}$$

Substituting f'' from (6) in (7) we obtain the following integral for ϕ:

$$\phi = \pm\sqrt{2} \int dr \sqrt{\Lambda - f'^2} \tag{8}$$

Thus if we solve (6) we can obtain both metric functions u_i and ϕ.

2.1 About the Stability of the Field Equations

Now we investigate the stability of the field equations (3, 4) under small perturbations. First we note that instead of work on the system (3, 4) we can treat the (6, 8) as the field equations in linear approximation (6) becomes

$$f'' \approx 3\Lambda$$

The general solution for this simple ODE is

$$f(r) = \frac{3\Lambda}{2} r^2 + c_1$$

If we take the de-Sitter radius as $a = \sqrt{\frac{3}{\Lambda}}$ this function is nothing but the usual metric function of the Asymptotic de-Sitter. If we want to check the eigenvalues of the linearized matrix for system of field equations we must change the system (3, 4) to a higher rank first order system. For this, we introduce the following set of new variables,

$$u_i = x_i i = \{1, 2, 3\}, \quad \phi = y$$

Then we have

$$2x_i' + x_i \sum_{j=1}^{3} x_j - 4\Lambda = 0, \quad i = \{1, 2, 3\}$$

$$2\sum_{j=1}^{3} x_j' + \sum_{j=1}^{3} x_j^2 - 4\Lambda = 4y'^2$$

For a stationary point in the phase space we must set all first order derivatives equal to zero,

$$x_i \sum_{j=1}^{3} x_j - 4\Lambda = 0, \quad i = \{1, 2, 3\}$$

$$\sum_{j=1}^{3} x_j^2 - 4\Lambda = 0, \quad y = 0$$

In the first equation if we summing on the indices and comparing both of them we obtain

$$x_i = 2/a, \quad i = \{1, 2, 3\}, \ y = 0$$

We perturb the field equations as

$$x_i = 2/a + \delta x_i, \quad i = \{1, 2, 3\}, \quad y = \delta y$$

Expanding the field equation in these perturbations up to first order and taking mind to having an asymptotic stability the real part of the eigenvalues must be negative. In the case of our field equations the matrix poses negative purely real eigenvalues, thus according to the Lyapunov theorem this system is stable.

3 Exact Solutions

In this section we investigate all possible solutions for (6), (8).

The exact solution for ordinary differential equation (6) is:[3]

$$f(r) = -\sqrt{3\Lambda}r + \frac{1}{2} \log\left(\frac{1}{12\Lambda} \left(c_1 e^{2\sqrt{3\Lambda}r} - c_2 \right)^2 \right) \tag{9}$$

We keep the relation (9) as a constraint for all solutions of (6). Substituting this function in (5) we can write the following general form of metric functions:

$$u_i = -\frac{\alpha_i}{3} \sqrt{\frac{3}{\Lambda}} \frac{1}{\sqrt{c_1 c_2}} \tanh^{-1}\left(\sqrt{\frac{c_1}{c_2}} e^{\sqrt{3\Lambda}r} \right) + \frac{2}{3} \log\left(\frac{c_1 e^{2\sqrt{3\Lambda}r} - c_2}{e^{\sqrt{3\Lambda}r}} \right) + \beta_i, \quad i = 1, 2, 3 \tag{10}$$

here we note that if the solutions (10) satisfy in (9), then we have $\sum_{j=1}^{3} \alpha_j = 0$. Thus the final form of metric functions in:

$$u_i = \frac{2}{3} \log\left(\frac{c_1 e^{2\sqrt{3\Lambda}r} - c_2}{e^{\sqrt{3\Lambda}r}} \right) + \beta_i, \quad i = 1, 2, 3 \tag{11}$$

Later we will determine the coefficients c_1, c_2. The condition $\sum_{j=1}^{3} \beta_j = -\frac{1}{2} \log(12\Lambda)$ may be satisfied by adding a constant β_j to each metric function which can be absorb in the non radial coordinates. In the languages of potential theory we can choose a new gauge for potential functions $u_i(r)$. The reason is that in cylindrical spacetimes we know that the solving of the Einstein field equations can be reduced to solving a system of potential equations [7]. A better option is the comparison of (11) with the previous metric function in [1]:

$$u_i = u(r) = \pm \sqrt{\frac{\Lambda}{3}} r + \frac{2}{3} \log(1 + \xi^2 e^{2\sqrt{3\Lambda}r}), \quad i = 1, 2, 3 \tag{12}$$

As it was attend that in [1] by the authors, only the minus sign in metric function is consistent with scalar field equation of motion $\Box\phi = 0$. In terms of a new parameter $\xi = -i\sqrt{\frac{c_1}{c_2}} \in \mathbb{R}$—which was defined in [1]—we must choose the constant β_j as the following:

$$\beta_j = -\frac{2}{3} \log(-c_2), \quad i = 1, 2, 3 \tag{13}$$

Obviously if $c_1 \in \mathbb{R}$, then $c_2 \in \mathbb{C}$ (purely imaginary) and consequently β_j is real.

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[3] $\log(x) = \int_1^x \frac{d\zeta}{\zeta}$.

4 Validity of Energy Condition in LB Metric

The origin of the null energy condition (NEC) and of the strong energy conditions (*SEC*) is the Raychaudhuri equation together with the requirement that the gravity is attractive for a spacetime manifold endowed with a metric $g_{\hat{\mu}\hat{\nu}}$ [46].[4]

The classical energy conditions of general relativity, to the extent that one believes that they are a useful guide [13, 14], allow one to deduce physical constraints on the behavior of matter fields in strong gravitational fields or cosmological geometries. These conditions can most easily be stated in terms of the components of the stress energy tensor $T^{\hat{\mu}\hat{\nu}}$ in an ortho-normal frame. Ultimately, however, constraints on the stress-energy are converted, via the Einstein equations, to constraints on the spacetime geometry.[5] For a perfect fluid cosmology, and in terms of pressure and density, the so-called *Null, Weak, Strong and Dominant energy* conditions reduce to [15]:

NEC: $p_i + \rho \geq 0$

WEC: This specializes to the *NEC* plus $\rho \geq 0$

SEC: This specializes to the NEC plus $\sum p_i + \rho \geq 0$

DEC: $\rho \geq |p_i|$.

Note particularly that in FRW models of Universe, the condition *SEC* is independent of the space extrinsic curvature k. Now, *DEC* implies *WEC* implies *NEC*, and SEC implies *NEC*, but otherwise the *NEC*, *WEC*, *SEC*, and *DEC* are mathematically independent assumptions. In particular, the SEC does not imply the *WEC*. Violating the *NEC* implies violating the *DEC*, *SEC*, and *WEC* as well [15]. Note that ideal relativistic fluids satisfy the *DEC*, and certainly all the known forms of normal matter encountered in our solar system satisfy the *DEC*. With sufficiently strong self-intereactions relativistic fluids can be made to violate the *SEC* (and *DEC*); but classical relativistic fluids always seem to satisfy the *NEC*. Most classical fields (apart from non-minimally coupled scalars) satisfy the *NEC*. Violating the *NEC* seems to require either quantum physics (which is unlikely to be a major contributor to the overall cosmological evolution of the universe) or non-minimally coupled scalar fields (implying that one is effectively adopting some form of scalar-tensor gravity). Using this dynamical formulation of the energy conditions, Santos et al. [16] derive some bounds, for the special case $k = 0$ (flat cosmology), on the luminosity distance d_L of supernovae, and then contrast this with the legacy [17, 18] and gold [19] datasets. In Ref. [20] bounds on the distance modulus are presented for general values of k while in Ref. [21] they concentrate on the lookback time.

Due to the lack of satisfactory dark energy models, many model-independent methods were proposed to study the properties of dark energy and the geometry of the universe [22–45]. Another very interesting and model-independent approach is to consider the energy conditions [46, 47]. Recently Energy Conditions and Stability in $f(R)$ theories of gravity with non-minimal coupling to matter is discussed and determined the bounds from the energy conditions on a general $f(R)$ functional form in the framework of metric variational approach [48, 49].

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[4]This is a Tetrad's representation of the metric.

[5]In particular in a FRW spacetime one is ultimately imposing conditions on the scale factor and its time derivatives.

5 Checking the Focusing Theorem

In this section first we state the focusing theorem in the manner of congruences both in time like and null cases. Then we checked that our metric which is constructed from a messless cylindrically symmetric scalar field in the presence of a cosmological constant satisfies this theorem or not?

5.1 Focusing Theorem in General Relativity

Let a congruence of time like geodesics be hypersurface orthogonal, means there is exist a vector field u^α (time like, space like or null and not necessarily geodesic). We called it a hypersurface orthogonal if $\omega_{\alpha\beta} \equiv u_{[\alpha;\beta]} = 0$. It means that there exist a scalar field Φ such that $u_\alpha \propto \Phi_{,\alpha}$ and let the (SEC) hold. So that from Einstein field equations

$$R_{\alpha\beta} u^\alpha u^\beta \geq 0.$$

From Raychaudhuri equation (8) implies:

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - R_{\alpha\beta}u^\alpha u^\beta \leq 0 \tag{14}$$

where in it, $\theta = u^\alpha_{;\alpha} = B^\alpha_\alpha$ is the expansion scalar, $\sigma_{\alpha\beta} = B_{(\alpha\beta)} - \frac{1}{3}\theta h_{\alpha\beta}$ the shear tensor, $h_{\alpha\beta}$ the transverse part of $g_{\alpha\beta}$ [9, 10] (which is purely 'spatial'). The expansion must therefor decrease during the congruence's evaluation. Focusing theorem stated that *an initially diverging ($\theta > 0$) congruence will diverge less rapidly in the future, while an initially converging ($\theta < 0$) congruence will converge more rapidly in the future*. The physical interpretation is that gravitation is an attractive force when the (SEC) holds and the geodesics get focused as a result of this attraction.

For a congruence of null geodesics be hypersurface orthogonal and (NEC) hold, too we have $\frac{d\theta}{d\tau} \leq 0$ where $\theta = k^\alpha_{;\alpha}$ and k_α is the tangent vector field. It is important to realize that the Raychaudhuri equation is purely geometric and independent of the gravity theory under consideration. The connection with the gravity theory comes from the fact that, in order to relate the expansion variation with the energy-momentum tensor, one needs the field equations to obtain the Ricci tensor. Thus, through the combination of the field equations and the Raychaudhuri equation, one can set physical conditions for the energy-momentum tensor. The requirement that gravity is attractive imposes constraints on the energy-momentum tensors and establishes which ones are compatible. Of course, this requirement may not hold at all instances. Indeed, a repulsive interaction is what is needed to avoid singularities as well as to achieve inflationary conditions, and to account the observed accelerated expansion of the universe.

5.2 Straightforward Calculations

Now, we consider a congruence of radial, marginally bound, time like geodesics of the LB metric [1]:

$$ds^2 = dr^2 + w(r)(-dt^2 + d\varphi^2 + dz^2) \tag{15}$$

where,

$$w(r) = e^{-2\sqrt{\frac{\Lambda}{3}}r}(\xi^2 e^{2\sqrt{3\Lambda}r} + 1)^{2/3}$$

For radial geodesics,the components of 4-vector field u^α, $u^\varphi = u^z = 0$, and the geodesics are marginally bound if $-u_\alpha \xi^\alpha_{(t)} = -u_t = \tilde{E}$. This means that the conserved energy is precisely equal to the rest-mass energy, and this gives us the equation $u^t = \frac{\tilde{E}}{w(r)}$.[6] From LB metric [1] we know that $w(r) > 1$. Indeed since the radial like component of 4-vector velocity must be a real function it can deduced that

$$|\tilde{E}| \geq 1$$

by the normalization condition

$$g_{\alpha\beta} u^\alpha u^\beta = -1$$

we have:

$$u^r = \pm \sqrt{\frac{\tilde{E}^2}{w(r)} - 1}$$

The upper sign applies to outgoing geodesics, and the lower sign applies to ingoing geodesics. The 4-velocity is given by:

$$u^\alpha = \left(\frac{\tilde{E}}{w(r)}, \pm \sqrt{\frac{\tilde{E}^2}{w(r)} - 1}, 0, 0 \right) \tag{16}$$

by using (16) we can write:

$$u^\alpha \partial_\alpha = \frac{1}{w(r)} \partial_t \pm \left(\frac{\tilde{E}^2}{w(r)} - 1 \right)^{1/2} \partial_r$$

$$u_\alpha dx^\alpha = -\tilde{E} dt \pm \left(\frac{\tilde{E}^2}{w(r)} - 1 \right)^{1/2} dr$$

It follows that u_α is equal to a gradient of a scalar function Φ where:

$$u_\alpha = -\Phi_{,\alpha}$$

and:

$$\Phi = -t \pm \int \left(\frac{\tilde{E}^2}{w(r)} - 1 \right)^{1/2} dr \tag{17}$$

The integral could be written in terms of hypergeometric functions which we don't write it here. This expression means that the congruence is everywhere orthogonal to the spacelike hypersurfaces $\Phi = constant$. The expansion is calculated as:

$$\theta = u^\alpha_{;\alpha} = \pm w(r)^{-3/2} \frac{d}{dr} \left(w(r) \sqrt{\tilde{E}^2 - w(r)} \right) \tag{18}$$

Not surprisingly, the congruence is *diverging* ($\theta > 0$) if the geodesics are *outgoing*, and *converging* ($\theta < 0$) if the geodesics are *ingoing*. The rate of change of the expansion is

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[6]Notice that only in Schwarzschild spacetime (which has the property $g_{tt} g_{rr} = -1$)the choose of $\tilde{E} = 1$ is a good coordinate's representation and in general ,specially in the form of a cylindrically symmetric LB metric, we must keep $\tilde{E} \neq 1$ to avoiding of occurrence an unphysical radial velocity.

calculated as:

$$\frac{d\theta}{d\tau} = \left(\frac{d\theta}{dr}\right) \cdot \frac{dr}{d\tau} = \acute{\theta} u^r$$

and the result is:

$$\frac{d\theta}{d\tau} = \left(\frac{\tilde{E}^2}{w(r)} - 1\right)^{\frac{1}{2}} \frac{d}{dr}\left[w(r)^{-3/2} \frac{d}{dr}\left(w(r)\sqrt{\tilde{E}^2 - w(r)}\right)\right] \tag{19}$$

Substituting $w(r)$ and performing differentiation, in terms of a new variables

$$x \equiv \frac{w(r)}{\tilde{E}^2} \in (\sqrt[3]{4b^2}, 1), \quad |b| < \frac{1}{2}$$

$$b \equiv \left|\frac{\xi}{\tilde{E}}\right|$$

$$y \equiv \sqrt{x^6 - 4b^2 x^3}$$

$$a = \sqrt{\frac{3}{\Lambda}}$$

We have:

$$\frac{d\theta}{d\tau} = \frac{\Lambda}{2} \frac{\Phi_b(x, y)}{x(1-x)} \tag{20}$$

where,

$$\Phi_b(x, y) = \frac{(27x^2 - 45x + 20)y - 36b^2x^2 - 46x^4 + 27x^5 + 64b^2x + 20x^3 - 32b^2}{3(x^3 + y)} \tag{21}$$

This polynomial has at most two real roots, as can be seen from the fact that its second derivative is always positive and it vanishes whenever there is a double root. As mentioned before, in order to ensure reality, the root point must be at the right of the outer region $w(r) < 1$. For $|b| < 1/2, 0 < x < 1$ this function always poss negative values. For $\frac{d\theta}{d\tau} = 0$ this entails to look for an stationary point of the equation $\Phi_b(x, y) = 0$. This equation can not be solved. But for suitable values of b this polynomial in x has a second derivative that is everywhere positive regardless of the value of b, implying that it may have at most two real roots that can be identified with two *event horizons*. If one take $b = 0$, when the minus sign is chosen, the metric (15) corresponding to the $LC\Lambda$ case already analyzed in the previous work [1]. These choose will coincide when the roots become $w(r) = 0.377, 1.178$ and a single double root $w(r) = 0$ (which is not accessible), by $w(r) = 1.178$ we have

$$r = 0.0273a$$

For a congruence of null geodesics which must be hypersurface orthogonal (in the manner that is discussed in part(a)) and (NEC) hold, we must have $\frac{d\theta}{d\lambda} \leq 0$ where $\theta = k^\alpha_{;\alpha}$ where k_α is the tangent vector field.

For $d\varphi = dz = 0$ the (LB) line element reduces to:

$$ds^2 = -w(r)\left(dt - \frac{dr}{\sqrt{w(r)}}\right)\left(dt + \frac{dr}{\sqrt{w(r)}}\right)$$

The displacements will be *null* if $ds^2 = 0$. We define two null coordinates,

$$u = t - r^*$$
$$v = t + r^*$$

which as usual we introduced a *tortoise coordinate*[7]

$$r^* = \int \frac{dr}{\sqrt{w(r)}} = a e^{\frac{r}{a}} F\left(\frac{1}{6}, \frac{1}{3}, \frac{7}{6}, -\xi^2 e^{\frac{6r}{a}}\right)$$

Easily we find that on *out-going* null geodesics $u = constant$ and similarly $v = constant$ on *in-going* ones. The following vectors are *null*,

$$k_\alpha^{out} = -\partial_\alpha u$$
$$k_\alpha^{in} = -\partial_\alpha v$$

They both satisfies the geodesic equation with $+r$ as an affine parameter for k_α^{out} and $-r$ for k_α^{in}. The congruences are clearly hypersurface orthogonal. Expansion(s) are calculated:

$$\frac{d\theta}{d\lambda} = \frac{1}{w(r)} \sqrt{\tilde{E}^2 - w(r)} \left[w''(r) - \frac{3}{2} \frac{w'(r)^2}{w(r)} \right] \tag{22}$$

This function never vanishes and remains always negative. We can construct a similar function as (15) for it and determining the sign of it. Another simple method is drawing a graph for $\frac{d\theta}{d\lambda}$. Appling any of this two methods prove that this function is negative every where.

6 Summary

In this short report we found the unique exact solution for field equations containing a massless scalar field and a cosmological constant term. We checked the stability, also we showed that there is some restriction on the energy conditions. This exact solution satisfy energy conditions and the validity of focusing theorem proved directly.

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[7] $F(a, b, c; z) = \sum_{k=0}^\infty \frac{(a)_k (b)_k}{(c)_k \Gamma(k+1)} z^k, (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}, \Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt.$

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